then either the final m or final n (or both) should be increased. This way, the size of the originally small determinant is expanded to a still small, but acceptably accurate, determinant. Note that contraction also is possible by the same procedure. Thus, by employing this procedure, with the solution of a few reduced-order matrices, we can zero in to the final acceptable set: in this case, the $m=1$-4 and $n=20$-24 determinant. The computer time required for accurately predicting k_s by solving several times reduced-size matrices is much smaller than the time required to solve the large-order matrices. This conclusion is reinforced by the fact that, in most problems, the position of the most influencing term is never known a priori. This implies that, if we want to predict k_s accurately by solving a single determinant, its size must be extremely large. The computational procedure proposed to handle all problems containing shear is outlined below (fully automated):

1) Choose the range of m and n according to some a priori knowledge concerning the effects of different geometric parameters. This is facilitated by the parametric study reported herein.

2) Compute k_s, X_1, and X_2 for the preceding range of m and n by employing Eqs. (4) and (6).

3) Check the elements of the eigenvectors X_1 and X_2, and generate the new range of m and n so as to zero in to a final set that includes only elements that influence the buckling mode. (The 10% criterion is used here.)

4) Once this final range is located, compute k_s by Eq. (4). This value is taken as the accurate prediction for the critical condition (output).

The proposed computational procedure is employed in performing a number of parametric studies. The computations were made with the aid of the Georgia Tech high-speed digital computer UNIVAC 1108. A systematic program of study was undertaken to assess 1) the effectiveness of the procedure, and 2) the effect of a number of geometric parameters on the needed range of m and n to predict accurately critical conditions for stiffened or unstiffened configurations under destabilizing loads that include shear. These parameters are the panel aspect ratio ξ, the curvature parameter Z and the contribution to the extensional and flexural stiffness of panel made by the stiffeners (four parameters). A large range was chosen for these parameters which enhances the generality of the conclusions. Only some of the generated data are presented in tabular form here. The conclusions are based on all generated data. In Table 1 the value of $k_{s_{cr}}$, the most influencing terms, and the computer CPU time in milliseconds are given for a number of m and n ranges for an unstiffened, thin, circular panel. The geometric parameters are given on the table. (The symbols are the same as those in Ref. 5.) The reported computer time is the total time required for both symmetric and antisymmetric $k_{s_{cr}}$ calculations. Similar results are presented in Tables 2 and 3. Table 2 deals with the case of an unstiffened, thin, circular panel under the combined action of shear and uniaxial compression (k_x) (an example taken from Ref. 2). Table 3 deals with a stiffened, circular panel under shear only. This example is taken from Ref. 1. The results presented on these three tables clearly demonstrate the effectiveness of the proposed method.

Among the most important conclusions of the present investigation, one may list the following:

1) A fully automated computational procedure is developed which accurately predicts critical conditions for stiffened and unstiffened thin circular panels under the action of destabilizing loads, which include shear, with reasonable computational time.

2) For both stiffened and unstiffened panels, the effect of both the panel aspect ratio ξ and the curvature parameter Z on the needed size of matrices (determinant) for accurately predicting critical conditions that include shear is significant.

3) Conversely from the preceding conclusion, the effect of the amount and type of stiffening is rather insignificant. This conclusion is based on the large amount of generated data.

These last two conclusions are very important in the optimization of stiffened panels under destabilizing loads that include shear, because they provide a means for the investigator to generate a large amount of data in the design space with relatively small computer time (see Ref. 5).

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Nonlinear Vibrations of Beams Considering Shear Deformation and Rotary Inertia

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Introduction

LARGE-amplitude free vibrations of slender beams have been analyzed by use of continuum[1,2] and finite-element[3] methods. The purpose of the present note is to study the effect of shear deformation and rotatory inertia on the large-amplitude vibrations of beams. The method followed is based on an appropriate linearization of the nonlinear strain-displacement relations. The linearized stiffness and mass matrices are derived using standard principles.[4] The resulting linear algebraic eigenvalue problem is solved by employing an iterative technique[3] to obtain the nonlinear frequencies. In this paper, both simply supported and clamped beams are considered. The ratios of nonlinear frequency (ω_{NL}) to linear frequency (ω_L) for the fundamental mode are obtained for various values of slenderness ratios and central amplitude ratios.

Finite-Element Formulation

The nonlinear strain-displacement relations of a beam including shear deformation are given by[5]

$$\epsilon_x = \tfrac{1}{2}(\mathrm{d}w/\mathrm{d}x)^2, \quad \psi_x = -[(\mathrm{d}^2w/\mathrm{d}x^2) + (\mathrm{d}\gamma/\mathrm{d}x)] \quad \text{and}$$

$$\epsilon_{xz} = -\chi\epsilon_{xz} \qquad (1)$$

where ϵ_x and ψ_x are the direct strain and curvature, respectively, ϵ_{xz} is the shear strain, w is the lateral displacement, γ is the shear rotation, and x is the axial coordinate.

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Table 1 Convergence of ω_{NL}/ω_L for a simply supported beam (fundamental mode)

a/r	$\ell/r=100$			$\ell/r=10$		
	2 elements	4 elements	8 elements	2 elements	4 elements	8 elements
1.0	1.08890	1.08900	1.08900	1.11232	1.11164	1.11162
2.0	1.31227	1.31249	1.31249	1.37882	1.37058	1.37006
3.0	1.60265	1.60295	1.60295	1.71710	1.69149	1.68839

Table 2 Convergence of ω_{NL}/ω_L for a clamped beam (fundamental mode)

a/r	$\ell/r=100$			$\ell/r=10$		
	4 elements	8 elements	16 elements	8 elements	12 elements	16 elements
1.0	1.02190	1.02182	1.02179	1.04370	1.04180	1.04094
2.0	1.08390	1.08353	1.08340	1.16002	1.15327	1.15010
3.0	1.17722	1.17624	1.17592	1.32377	1.31084	1.30466

Table 3 ω_{NL}/ω_L of a simply supported beam (fundamental mode)

a/r	$\ell/r=10$	$\ell/r=25$	$\ell/r=50$	$\ell/r=100$	For slender beams without considering shear deformation and rotary inertia Ref. 1	Ref. 2	Ref. 3
0.4	1.01928(4)[a]	1.01558 (3)	1.01504 (3)	1.01491 (3)	...	...	...
0.8	1.07366 (6)	1.06047 (5)	1.05854 (4)	1.05805 (4)	...	...	...
1.0	1.11162 (7)	1.09253 (5)	1.08971 (5)	1.08900 (5)	1.0892	1.0897	1.0844
2.0	1.37006 (16)	1.32130 (8)	1.31425 (7)	1.31249 (7)	1.3178	1.3220	1.3033
3.0	1.68839 (47)	1.61464 (14)	1.60520 (11)	1.60295 (11)	1.6257	1.6370	1.5997

[a]Numbers in parentheses indicate the number of iterations required to achieve an accuracy of 10^{-6} in the evaluation of ω_{NL}/ω_L.

Table 4 ω_{NL}/ω_L of a clamped beam (fundamental mode)

a/r	$\ell/r=10$	$\ell/r=25$	$\ell/r=50$	$\ell/r=100$	For slender beams without considering the effect of shear deformation and rotary inertia (Ref. 3)[b]
0.4	1.00676 (3)[a]	1.00387 (3)	1.00359 (3)	1.00353 (3)	...
0.8	1.02654 (5)	1.01533 (4)	1.01425 (4)	1.01403 (3)	...
1.1	1.04094 (5)	1.02376 (4)	1.02213 (4)	1.02179 (4)	1.02
2.0	1.15010 (10)	1.08950 (6)	1.08436 (5)	1.08340 (4)	1.08
3.0	1.30466 (30)	1.18552 (7)	1.17706 (7)	1.17592 (6)	...

[a]Numbers in parentheses indicate the number of iterations required to achieve an accuracy of 10^{-6} in the evaluation of ω_{NL}/ω_L.
[b]Values read from graph.

The nonlinear strain-displacement relations given in Eq. (1) are linearized as

$$\epsilon_x = f(\mathrm{d}w/\mathrm{d}x) \quad \psi_x = -[(\mathrm{d}^2 w/\mathrm{d}x^2) + (\mathrm{d}\gamma/\mathrm{d}x)] \text{ and}$$

$$\epsilon_{xz} = -\gamma \tag{2}$$

where f is the linearizing function given by ½ $(\mathrm{d}w/\mathrm{d}x)$.

The linearized strain energy U, including shear deformation, of the beam element, is given by

$$U = ½\int_0^{\ell_e} [EI\psi_x^2 + EA\epsilon_x^2 + k\{EA/2(1+\nu)\}\epsilon_{xz}^2]\,\mathrm{d}x \tag{3}$$

where k is the shear coefficient (in the present study k is assumed as 5/6) E is the Young's modulus, I is the moment of inertia, A is the area of cross section, ν is the Poisson's ratio (assumed as 0.3), and l_e is the element length.

The kinetic energy T, including rotatory inertia of the beam element, executing harmonic oscillations, is given by

$$T = \frac{m\omega^2}{2}\int_0^{l_e} \left[w^2 + r^2\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \gamma\right)^2\right]\mathrm{d}x \tag{4}$$

where, m is the mass per unit length, ω is the circular frequency, and r is the radius of gyration.

The displacement distributions for w and γ over the beam element are assumed as

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

and

$$\gamma = \alpha_5 + \alpha_6 x \tag{5}$$

Using classical displacement method,[4] the linearized stiffness and mass matrices of the beam element are obtained (the nodal parameters being $w, dw/dx$, and γ). The resulting linear algebraic eigenvalue problem is solved by employing an iterative technique[3] to obtain nonlinear frequencies.

Numerical Results and Discussion

In the present study, the ratio of nonlinear frequencies (ω_{NL}) to the linear frequencies (ω_L) of simply supported and clamped beams, for the fundamental mode, are obtained for various l/r and a/r ratios, where l is the length of the beam and a is the central amplitude. It may be pointed out here that, in all cases, for an n element idealization, the frequency ratios are calculated by dividing ω_{NL} by ω_L, both obtained with the same element idealization.

Tables 1 and 2 present the convergence study for simply supported and clamped beams, respectively, for $l/r = 100$ and 10, and for $a/r = 1.0$, 2.0, and 3.0. It is seen from these tables that, for a simply supported beam an 8-element idealization, and for a clamped beam a 16-element idealization in the half of the beam, gave satisfactory convergence of the ratio ω_{NL}/ω_L. (The rate of convergence can be improved further by assuming a cubic polynomial for γ, also in the displacement distribution.) Hence, for all further calculations, these element idealizations are used.

Tables 3 and 4 give the ratios of ω_{NL}/ω_L of simply supported and clamped beams, respectively, for various l/r and a/r ratios. From these tables it is observed that the effect of shear deformation and rotatory inertia is to increase the nonlinearity, and this effect increases as l/r decreases. For slender beams (for example $l/r = 100$), the results obtained by the present formulation agree well with the results, which also are included in these tables, reported in Ref. 1-3.

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An Experiment on the Lift of an Accelerated Airfoil

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Introduction

THIS Note presents the results of an experiment to study the lift characteristics of a two-dimensional airfoil undergoing very high accelerations. The experimental data were obtained by mounting a stationary airfoil in the driver section of a simple shock tube, and utilizing the high fluid accelerations which occur in an expansion fan. In performing the experiment with the airfoil stationary and the gas accelerating, we made use of this result: that the flow and forces which act on a stationary airfoil in an accelerating flow are the same as those on an accelerating airfoil, except for the bouyancy force due to the fluid acceleration which is small, and is orthogonal to the lift. This equivalence between unsteady motion of a body and unsteady motion of a fluid stream about a stationary body appears to be made quite frequently in unsteady flow problems.[1]

The motivation for this experiment arose from work associated with the flight of small insects, which appear to require anomalously high values of the lift coefficient to sustain flight. In discussing this effect, Osborne[2] concluded that the large lift coefficients obtained for insect flight must be attributed to acceleration effects. Recently Weis-Fogh[3] has indicated that nonsteady aerodynamics must play a major role in the flight of very small insects. We have attempted to display the influence of acceleration effects in Fig. 1, which shows a plot of minimum lift coefficient required to support the insect weight plotted against a parameter $\dot{U}c/U^2$, which we call the relative acceleration parameter. These data were taken from Osborne's data for 25 different insects. The velocity U was taken as the vector sum of the flight velocity and the flapping vel., and both U and $\dot{U}$ were computed at a radius equal to 0.75 of the wing length, assuming sinusoidal motion of the wing. The chord length c was computed from the given wing area and length assuming rectangular shape. The final values of $\dot{U}c/U^2$ were computed from an average of 10 positions over a quarter cycle taken from the top of the stroke.

It can be seen from Fig. 1 that the insect lift coefficient increases with increasing values of $\dot{U}c/U^2$. This led us to consider the simple experiment described herein, in which the effect of airfoil acceleration on lift is isolated and evaluated. The experiments were performed at comparable values of $\dot{U}c/U^2$ to those shown for insects in Fig. 1. However, it is clear that our experiment does not model insect flight, since the particular time history of the flapping motion is not reproduced. Furthermore, other features of insect wings such as wing shape, surface properties, wing rotation and bending, all of which have been suggested as possible explanations of high lift, are not included in our experiment.

Experiments

The experiments were performed in a shock tube of square cross section with an internal side length of 3 in. Air was used for both the high and low pressure gases. All unsteady pressures were recorded as functions of time, using Kistler model 601A pressure transducers and model 566 charge amplifiers.

The airfoil model was a NACA 0015 symmetric profile with a span of 3.0 in. and a chord of 0.5 in. Five identical airfoils were manufactured, each mounted at a fixed angle of attack (10°, 14°, 25°, 30°,40°) to a short cylindrical sting. The airfoils were mounted so that positive lift forces were downwards. The sting was placed through a linear ball bushing mounted on the bottom of the shock tube, and was fitted with a collar to prevent motion out of the vertical direction. The end of the sting rested on a thin nylon cap, which in turn rested on the head of a Kistler pressure transducer. With this arrangement, lift forces were recorded in the same way as pressure forces. All force and pressure measurements were made at a location 1 ft from the diaphragm position.

The output of the lift-measuring transducer was corrected to account for the fact that part of the transducer diaphragm was exposed to the local pressure in the rarefaction wave. Correction curves were obtained by running experiments using the sting without an attached airfoil, for initial diaphragm pressure ratios, P, of 2.0, 2.5 and 3.0. These 3 diaphragm pressure ratios were used for all the experiments with the airfoils.

Steady-state values of the lift coefficient were measured in order to provide a reference for the unsteady flow values. This was accomplished by opening both ends of the shock tube and installing a small blower at one end.

Results

The gas particle velocity in a simple rarefaction wave is given in terms of the local pressure by

$$U = \frac{2a_o}{\gamma - 1}\left[1 - (p/p_o)\ \exp\ (\gamma - 1)/2\gamma\right] \tag{1}$$

where p_o and a_0 are, respectively, the pressure and speed of sound in the initially undisturbed high pressure gas. The par-

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